EMD-Based Signal Filtering
Abdel-Ouahab Boudraa, Senior Member, IEEE, and Jean-Christophe Cexus

Abstract—In this paper, a signal-filtering method based on empirical mode decomposition is proposed. The filtering method is a fully data-driven approach. A noisy signal is adaptively decomposed into intrinsic oscillatory components called intrinsic mode functions (IMFs) by means of an algorithm referred to as a sifting process. The basic principle of the method is to make use of partial reconstructions of the signal, with the relevant IMFs corresponding to the most important structures of the signal (low-frequency components). A criterion is proposed to determine the IMF, after which, the energy distribution of the important structures of the signal overcomes that of the noise and that of the high-frequency components of the signal. The method is illustrated on simulated and real data, and the results are compared to well-known filtering methods. The study is limited to signals that were corrupted by additive white Gaussian noise and is conducted on the basis of extended numerical experiments.

Index Terms—Empirical mode decomposition (EMD), nonstationary signals, signal filtering.

I. INTRODUCTION

THE RECOVERY of a signal from observed noisy data, while preserving its important features (e.g., smoothness), remains a challenging problem in both signal processing and statistics. A number of filtering methods have been proposed, particularly for the case of additive white Gaussian noise (AWGN) [1]–[3]. Frequently, linear methods such as the Wiener filtering [1] are used because linear filters are easy to design and implement. However, linear filtering methods are not very effective when signals contain sharp edges and impulses of short duration. Furthermore, real signals are often nonstationary. To overcome these shortcomings, nonlinear methods have been proposed, particularly those that are based on wavelet thresholding [2], [3]. The idea of wavelet thresholding relies on the assumption that signal magnitudes dominate the magnitudes of the noise in a wavelet representation so that wavelet coefficients can be set to zero if their magnitudes are less than a predetermined threshold [3]. A limit of the wavelet approach is that the basis functions are fixed and, thus, do not necessarily match all real signals. To avoid this problem, time–frequency atomic signal decomposition can be used [4], [5]. As for wavelet packets, if the dictionary is very large and rich with a collection of atomic waveforms, which are located on a much finer grid in the time–frequency space than wavelet and cosine packet tables, then it should be possible to represent a large class of real signals (for denoising, compression, and so on). In spite of this, the basis functions must be specified (Gabor functions, damped sinusoids, and so on).

Recently, a new data-driven technique, referred to as empirical mode decomposition (EMD), has been introduced by Huang et al. [6] for analyzing data from nonstationary and nonlinear processes. The EMD has received more attention in terms of applications [7]–[17] and interpretations [18], [19]. The major advantage of the EMD is that the basis functions are derived from the signal itself. Hence, the analysis is adaptive in contrast to the traditional methods where the basis functions are fixed. The EMD is based on the sequential extraction of the energy associated with various intrinsic time scales of the signal, starting from finer temporal scales (high-frequency modes) to coarser ones (low-frequency modes). The total sum of the intrinsic mode functions (IMFs) matches the signal very well and, therefore, ensures completeness [6]. In our earlier paper, we have shown that the EMD can be used for signal denoising [10]. The method reconstructs the signal with all the IMFs that were previously thresholded, as in wavelet analysis, or filtered [10]. In this paper, the filtering scheme relies on the basic idea that most of the structures of the signal are often concentrated on lower frequency components (last IMFs) and decrease toward high-frequency modes (first IMFs). Thus, the recovered signal is reconstructed with only a few IMFs that are signal dominated. Consequently, compared to the approach introduced in [10], no thresholding or filtering is required. The proposed filtering method is a fully data-driven approach.

II. EMD ALGORITHM

The EMD involves the decomposition of a given signal $x(t)$ into a series of IMFs through the sifting process, with each one having a distinct time scale [6]. The decomposition is based on the local time scale of the signal and yields adaptive basis functions. The EMD can be seen as a type of wavelet decomposition whose subbands are built up as needed to separate the different components of $x(t)$. Each IMF then replaces the detail signals of $x(t)$ at a certain scale or frequency band [18]. The EMD picks out the highest frequency oscillation that remains in $x(t)$. A function is an IMF if R1) either the number of extrema and the number of zero crossings are equal or differ at most by one, and R2) at any point, the mean value of the envelope defined by the local maxima and the envelope defined by the local minima is zero. Thus, locally, each IMF contains lower frequency oscillations than the one that was extracted before. The EMD does not use any predetermined filter or wavelet function, and thus, it is a fully data-driven method [6]. To be
successfully decomposed into IMFs, the signal \( x(t) \) must have at least two extrema: one minimum and one maximum. The sifting process involves five major steps:

Step 1) Fix \( \epsilon, i \leftarrow 1 \) (jth IMF).
Step 2) \( r_{j-1}(t) \leftarrow x(t) \) (residual).
Step 3) Extract the jth IMF:
   a) \( h_{j,i-1}(t) \leftarrow r_{j-1}(t), \ i \leftarrow 1 \) (i number of sifts).
   b) Extract the local maxima/minima of \( h_{j,i-1}(t) \).
   c) Compute the upper and lower envelopes \( U_{j,i-1}(t) \) and \( L_{j,i-1}(t) \) by interpolating, using cubic spline, the local maxima and minima of \( h_{j,i-1}(t) \), respectively.
   d) Compute the envelope mean: \( \mu_{j,i-1}(t) \leftarrow \frac{[U_{j,i-1}(t) + L_{j,i-1}(t)]}{2} \).
   e) Update: \( h_{j,i}(t) \leftarrow h_{j,i-1}(t) - \mu_{j,i-1}(t), \ i \leftarrow i + 1 \).
   f) Calculate the stopping criterion by \( SD(i) = \sum_{t=0}^{N} \frac{|h_{j,i-1}(t) - h_{j,i}(t)|^2}{|h_{j,i-1}(t)|^2} \).
   g) Repeat Steps b)-f) until \( SD(i) < \epsilon \), and then, put \( IMF_j(t) \leftarrow h_{j,i}(t) \) (jth IMF).
Step 4) Update the residual: \( r_j(t) \leftarrow r_{j-1}(t) - IMF_j(t) \).
Step 5) Repeat Step 3 with \( j \leftarrow j + 1 \) until the number of extrema in \( r_j(t) \) is < 2.

Here, \( N \) is the time duration. The “\( a \leftarrow b \)" arrow is an assignment or a substitution operation; it means that the value of the variable \( a \) is to be replaced by the current value of the variable \( b \).

The sifting is repeated several times (\( i \)) to get \( h \) to be a true IMF that fulfills requirements (R1) and (R2). The result of the sifting is that \( x(t) \) will be decomposed into IMF\(_j\)s, \( j = 1, \ldots, C \), and a residual \( r_C(t) \) given by

\[
x(t) = \sum_{j=1}^{C} IMF_j(t) + r_C(t) \tag{1}
\]

where \( C \) is the number of modes, which is automatically determined using the stopping criterion \( SD \) [Step 3-f)].

Thus, \( C \) is signal dependent. The sifting process has two effects: 1) It eliminates riding waves, and 2) it smoothes uneven amplitudes. To guarantee that the IMF components will retain enough physical sense of both amplitude and frequency modulations, we have to determine a criterion for the sifting process to stop. This is accomplished by limiting the size of the standard deviation \( SD \) computed from the two consecutive sifting results. Usually, \( SD \) is set between 0.2 and 0.3 [6].

### III. Filtering Approach

The filtering method relies on the basic idea that most of the important structures of the signal are often concentrated on the lower frequency ones (last IMFs) and decrease toward high-frequency modes (first IMFs). Thus, one can assume that, for many signal classes that are corrupted by white noise, the signal-to-noise ratio (SNR) is higher at low frequencies than at high ones. According to this idea, there will be a mode indexed by \( j_s \), after which, the energy distribution of the important structures of the signal overcomes that of the noise and that of the high-frequency components of the signal. This particular mode IMF\(_{j_s}\) allows us to retrieve the most important structures of the signal from its noisy version. The modes after IMF\(_{j_s}\) are dominated by the signal, whereas the previous modes are high-frequency component dominated. In the proposed method, high-frequency-dominated modes are set to zero (hard thresholding) and will not be used in the signal reconstruction. The signal is partially reconstructed. The problem of reducing noise is complicated and difficult because the noise level in the signal is unknown. Furthermore, analytical expressions of the signal IMFs are not available. Most of the important results on the EMD are all based on the empirically determined findings from numerical experiments [6], [18], [19]. Thus, the filtering problem is empirically studied.

Consider a deterministic signal \( y(t) \) corrupted by an AWGN as follows:

\[
x(t) = y(t) + z(t). \tag{2}
\]

From this observed signal \( x(t) \), the objective is to find an approximation \( \tilde{y}(t) \) to the original signal \( y(t) \) that minimizes the mean square error (MSE) given by

\[
MSE(y, \tilde{y}) = \frac{1}{N} \sum_{i=1}^{N} [y(t_i) - \tilde{y}(t_i)]^2 \tag{3}
\]

where \( y = [y(t_1), y(t_2), \ldots, y(t_N)]^T \), and \( \tilde{y} = [\tilde{y}(t_1), \tilde{y}(t_2), \ldots, \tilde{y}(t_N)]^T \). Here, \( N \) is the length of the signal. Other distortion measures such as the mean absolute error (MAE) can be used [3]. Then, \( x(t) \) is first decomposed using the EMD (1) into IMF\(_j\)s, \( j = 1, \ldots, C \), and a residual \( r_C(t) \), and finally, \( \tilde{y}(t) \) is reconstructed using \((C - k + 1)\)-selected IMFs, starting from \( k \) to \( C \), as follows:

\[
\tilde{y}_k(t) = \sum_{j=k}^{C} IMF_j(t) + r_C(t), \quad k = 2, \ldots, C. \tag{4}
\]

The aim of the EMD filtering is to find the index \( k = j_s \) that minimizes the MSE\((y, \tilde{y})\). In practice, the mse or the MAE cannot be calculated because the original signal \( y(t) \) is unknown. In this paper, we propose a distortion measure, called consecutive mse (CMSE), that does not require any knowledge of \( y(t) \). This quantity measures the squared Euclidean distance between two consecutive reconstructions of the signal. The CMSE is defined as follows:

\[
CMSE(\tilde{y}_k, \tilde{y}_{k+1}) \equiv \frac{1}{N} \sum_{i=1}^{N} [\tilde{y}_k(t_i) - \tilde{y}_{k+1}(t_i)]^2, \quad k = 1, \ldots, C - 1 \tag{5}
\]

\[
\frac{1}{N} \sum_{i=1}^{N} [IMF_k(t_i)]^2. \tag{6}
\]

Thus, according to (6), the CMSE is reduced to the energy of the \( k \)th IMF. It is also the classical empirical variance estimate of the IMF [if \( k = 1, \tilde{y}_k(t) = x(t) \)]. Finally, the index \( j_s \) is given by

\[
j_s = \arg \min_{1 \leq k \leq C - 1} \{CMSE(\tilde{y}_k, \tilde{y}_{k+1}) \} \tag{7}
\]
Fig. 1. Test signals with $N = 2048$.

where $\tilde{y}_k$ and $\tilde{y}_{k+1}$ are signals that are reconstructed starting from the IMFs that are indexed by $k$ and $(k + 1)$, respectively. The criterion CMSE allows for the identification of the IMF order where there is the first significant change in energy. This empirical fact is derived from extended numerical experiments.

Fig. 2. Noisy test signals (SNR = 2 dB; SNR = −9 dB for ECG).
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The EMD filtering consists of six different steps.
Step 1) Fix SD (usually between 0.2 and 0.3 [6]).
Step 2) Run the sifting to extract the IMF \( k = 1, \ldots, C - 1 \) of \( x(t) \).
Step 3) Compute \( \tilde{y}_k(t) \), for \( k = 1, \ldots, C - 1 \), using (4).
Step 4) Compute \( \text{CMSE}(\tilde{y}_k, \tilde{y}_{k+1}) \), for \( k = 1, \ldots, C - 1 \), using (6).
Step 5) Compute \( j_s \) using (7).
Step 6) Reconstruct \( \tilde{y}_{j_s}(t) \), which is the filtered signal, using (4).

IV. RESULTS AND DISCUSSIONS

To test the EMD filtering method, we have performed numerical simulations for four test signals: 1) “Doppler,” 2) “Blocks,” 3) “Bumps,” and 4) “Heavysine.” These were obtained using the WAVELAB software.\(^1\) The time duration of these signals is 1 s, and the size is \( N = 2048 \). The method is also tested on two real signals: 1) a biomedical signal, i.e., electrocardiogram (ECG), and 2) a real turbulent pressure signal derived from a fluid mechanics system. The “ECG” and the turbulent signal are collected using sampling frequencies of 1 and 20 kHz, respectively. MSE and SNR are calculated as the measures of efficiency of noise reduction. For synthesized signals, the variance of the AWGN is set so that the original SNR (before filtering) is maintained at 2 dB. The SNR of the “ECG” is \(-9\) dB. The original signals and the corresponding noisy versions are depicted in Figs. 1 and 2, respectively [10]. The only parameter used to run the EMD filtering is the stopping criterion \( SD \),

\(^1\)Available from Stanford Statistics Department, courtesy of D. L. Donoho and I. M. Johnstone.
TABLE I

<table>
<thead>
<tr>
<th>Signals</th>
<th>“Doppler”</th>
<th>“Blocks”</th>
<th>“Heavysine”</th>
<th>“Bumps”</th>
<th>“ECG”</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>10</td>
<td>11</td>
<td>9</td>
<td>10</td>
<td>13</td>
</tr>
<tr>
<td>J_s</td>
<td>5</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

which is set to 0.25 [6]. Fig. 3 shows a sequential extraction of local oscillations by the EMD of the “Blocks” signal [Fig. 1(b)] [10]. The EMD decomposed the “Blocks” signal into 11 IMFs and a residual. Only five IMFs and the residual are represented. One can remark that the first IMF corresponds to a fast oscillation, whereas the fifth IMF corresponds to a slow one (Fig. 3). A comparison of the signal (top diagram) and the residue (bottom diagram) in Fig. 3 shows that the residue captures the trend of the signal. For the remaining signals, we obtained similar decompositions as in the “Blocks” signal. Thus, for an illustration of the method, we have restricted ourselves to the “Blocks” signal. The C values used in the reconstruction of the different signals are reported in Table I. Note that for the same SD value, the C value is signal dependent. For the filtering scheme, each noisy signal is decomposed into IMFs, and the J = J_s index is calculated using the CMSE minimum value (7). The filtered signals are constructed according to (4), where the signal is partially reconstructed using IMFs ranging from J_s to C (Table I), and the corresponding SNRs are estimated (Fig. 4). Fig. 4 shows the plots of the SNR values versus the number of IMFs for the five signals. The SNR maximum of each signal occurs at the J_s value (Table I). Note that each SNR plot exhibits a single hump. This fact is derived from extended numerical experiments. Fig. 5 displays the outcome of applying the EMD filtering scheme to the five signals. Each reconstructed signal plot (solid line) is superposed on the corresponding

Table II

FILTERING RESULTS OF THE DIFFERENT SIGNALS THAT WERE CORRUPTED BY THE GAUSSIAN NOISE

<table>
<thead>
<tr>
<th></th>
<th>“Doppler”</th>
<th>“Blocks”</th>
<th>“Bumps”</th>
<th>“Heavysine”</th>
<th>“ECG”</th>
</tr>
</thead>
<tbody>
<tr>
<td>SNR</td>
<td>2.06</td>
<td>1.04</td>
<td>2.03</td>
<td>1.04</td>
<td>2.03</td>
</tr>
<tr>
<td>MSE</td>
<td>1.04</td>
<td>2.03</td>
<td>1.04</td>
<td>2.03</td>
<td>1.04</td>
</tr>
<tr>
<td></td>
<td>2.03</td>
<td>1.04</td>
<td>2.03</td>
<td>1.04</td>
<td>-9.02</td>
</tr>
<tr>
<td>Noisy</td>
<td>0.33</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Averaging</td>
<td>9.86</td>
<td>0.17</td>
<td>9.06</td>
<td>0.20</td>
<td>9.46</td>
</tr>
<tr>
<td></td>
<td>0.19</td>
<td>12.66</td>
<td>0.09</td>
<td>7.23</td>
<td>8x10^-3</td>
</tr>
<tr>
<td>Median</td>
<td>10.57</td>
<td>0.15</td>
<td>10.17</td>
<td>0.16</td>
<td>10.55</td>
</tr>
<tr>
<td></td>
<td>0.15</td>
<td>10.67</td>
<td>0.15</td>
<td>4.62</td>
<td>14x10^-3</td>
</tr>
<tr>
<td>Wavelet</td>
<td>14.97</td>
<td>0.05</td>
<td>11.94</td>
<td>0.10</td>
<td>14.47</td>
</tr>
<tr>
<td></td>
<td>0.06</td>
<td>18.76</td>
<td>0.02</td>
<td>5.82</td>
<td>11x10^-3</td>
</tr>
<tr>
<td>EMD</td>
<td>11.13</td>
<td>0.12</td>
<td>11.97</td>
<td>0.09</td>
<td>11.18</td>
</tr>
<tr>
<td></td>
<td>0.12</td>
<td>19.86</td>
<td>0.02</td>
<td>14.33</td>
<td>1.5x10^-3</td>
</tr>
</tbody>
</table>

Fig. 5. Results of the EMD filtering. Free-noise signals (dotted line). Reconstructed signals (solid line).
free-noise signal (dotted line). Globally, the results are qualitatively appealing; the reconstructions jump where the true signal jumps and are smooth where the true signal is smooth. The significant results are obtained for the “Blocks,” “Heavysine,” and “ECG” signals [Fig. 5(b), (d), and (e)], which are very close to the original signals. These findings are confirmed by the SNR values listed in Table II, where significant improvements in the SNR range from 9.07 to 23.35 dB. Table II shows a comparison of the mse and SNR values for the averaging, median, wavelet, and EMD methods. For the median and averaging methods, different window sizes (3, 5, 7, 9, 11, and 13) are tested, and only the best results from these two methods are reported in Table II. For the wavelet method, the symmlet of Daubechies of the fifth order is used, followed by the soft thresholding of Donoho [3]. As indicated in Table II, our filtering scheme outperforms the averaging and median methods. For the “Heavysine” and “ECG” signals, our method performs better than the wavelet approach. However, the wavelet method performs better than the EMD filtering for the “Doppler” and “Bumps” signals. The efficiency of the compared methods depends on the signal behavior. In particular, for the ECG signal, the averaging method achieves a better SNR than the wavelet method. A careful examination of the “Doppler” signal [Fig. 5(a)] shows that the EMD scheme failed to correctly reconstruct the left part of the signal. This is expected because this signal part corresponds to a high-frequency component (rapid frequency changing) that has been set to zero during the reconstruction process. The oscillations that we see in Fig. 5 are particularly in the vicinity of discontinuities and other rapid changes. These are "Gibbs-like" oscillations caused by the fact that the signals (solid lines) are the partial reconstructions obtained using only a reduced number of IMFs. It is important to keep in mind that using a white noise implies that its components are distributed over all IMFs. Thus, the partial reconstruction scheme of the signal removes the high-frequency components of both the noise and the signal and leaves the low-frequency components of the noise. Consequently, one expects that if only the noisy parts of the first IMFs are set to zero, this will damp some of the oscillations and, consequently, improve the performance of the method. The oscillations seen in the flat regions [Fig. 5(b) and (c)] may be due to the interpolation scheme that was used (cubic spline), and thus, it would be interesting to search for interpolation methods other than cubic splines. The filtering scheme is also tested on another real signal derived from a fluid mechanics system. The pressure signal is acquired from the transient rotational motion of a hydrofoil in a transitional (laminar to turbulent) flow regime. The signal is recorded from a transducer located in a cavity on the hydrofoil’s suction side [17]. The EMD results in 11 IMFs and a residue. Fig. 6 shows the noisy signal (dotted line) above which is plotted the partial reconstruction of the signal (solid line) with \( j_s = 10 \) along with the residual (dashed line). This representation clearly shows the EMD-filtering effect on the noisy signal. The obtained EMD-filtered signal from the partial reconstruction not only conserves the overall trend of the signal but also allows to superpose it on as many high-frequency modes (IMFs), as needed, to represent the signal. The resulting representation is more suitable for analysis, and the desired data is conserved.

**REFERENCES**


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